Lecture 7: Alex poly one knots are TOP slice.

- Help session May 22 @ 14-15h
- Next week: course survey!

History

1958: slice knots defined [Fox-Milnor]
1960s: algebraic concordance [Levine]
} originally only in smooth category

Why? Lack of "fundamental" results in TOP category.
e.g. ∃? normal bundles? transversality?

1965: even dim knots are slice [Kervaire]
1969: odd high dim knot conc roughly alg conc + exotic spheres [Levine]
} still either PL or smooth

see Teichner notes §3.3

1970s: Casson-Gordon invts "beyond alg conc" in classical dim.
1970s: Kirby-Siebenmann ⇒ high D results hold in TOP.
1980s: Breakthrough in 4D topology [Freedman-Quinn]

Landmark applications:

- 4D Poincaré conjecture: $M \simeq S^4 \Rightarrow M \approx S^4$ [Freedman, using constr. of Casson]
- "Fundamental tools" [Quinn]

- every map $\Sigma^2 \to M^4$ is htpic to a generic immersion
  i.e. loc. flat emb except fin. many transverse double pts.
- every loc. flat submfld of M^4 has a (linear) normal bundle
- Σ_1, Σ_2 loc. flat submflds of M^4. ∃ isotopy of M taking Σ_1 to Σ_1' such that Σ_1' and Σ_2 intersect transversely.

Many other applications: (TOP) h- /s- cob thms in dim 4
(TOP) surgery sequence exact (w. π_1 restriction)
(TOP) classification results for 4-mflds.
goal for today → Alex poly one knots are TOP slice

Note: once we have the "fund" tools, we note that alg conc. and CG invts ~~result~~ also obstruct TOP sliceness.

Sphere embedding thm [key technical result underpinning all of FQ]

M^4 closed, oriented, connected 4-mfld.
only for convenience

only TOP, but could say smooth for convenience.

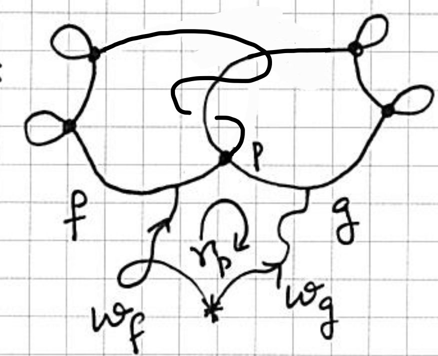
Recall: $\lambda := Q_M : H_2(M; \mathbb{Z}[\pi_1 M]) \times H_2(M; \mathbb{Z}[\pi_1 M]) \longrightarrow \mathbb{Z}[\pi_1 M]$

$H_2(M; \mathbb{Z}[\pi_1 M]) = \pi_2 M$, $H_2(M; \mathbb{Z}[\pi_1 M]) \cong \pi_2 M$ via Hurewicz.

the "equivariant intersection form"

Geometric interpretation: $f, g : S^2 \looparrowright M^4$ i.e. sm/top emb. except at fin. many. transv. dbl pts.

Schematic:



$$\lambda(f,g) := \sum_{p \in f \pitchfork g} \varepsilon_p \gamma(p) \in \mathbb{Z}[\pi_1 M]$$

equivariant int. "number"

Similarly,

$$\mu(f) := \sum_{q \in f \pitchfork f} \varepsilon_q \gamma(q) \in \mathbb{Z}[\pi_1 M] / g \sim g^{-1}$$

" equivariant self-int number

Note f embedding $\Rightarrow \mu(f) = 0$

<u>Thm</u>: Suppose $f, g : S^2 \looparrowright M$ s.t.
- $\mu(f) = 0$
- $\lambda(f,g) = 1$
- g has trivial normal bundle

And assume that $\pi_1 M$ is abelian or finite.

Then f is homotopic to a loc. flat emb $\bar{f}$

& g is homotopic to an immersion $\bar{g}$ s.t. $\bar{f} \pitchfork \bar{g} = pt$.

Note: algebra $\Rightarrow$ topology

in high D, $\mu(f) = 0 \Rightarrow f$ htpic to emb via Whitney trick.

$\pi_1 M$ can be more general. "Good" gp.

$\exists$ version for finite collections of $\{f, g\}$.

0-surgery characterisation of sliceness.

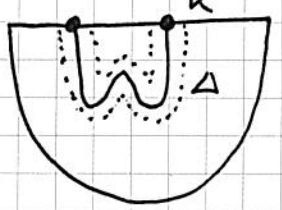
$K \subseteq S^3$ is TOP slice if and only if. $M_K := S^3_0(K) = \partial W^4$ for W compact, connected 4-mfld such that

(i) $\mathbb{Z} \cong H_1(M_K) \longrightarrow H_1(W)$ is an isomorphism

(ii) $\pi_1 W$ is normally generated by $\mu_K \subseteq M_K$. (i.e. gen by conjugates of μ_K)

(iii) $H_2(W) = 0$.

Proof: ($\Longrightarrow$)



Let $W := B^4 \setminus \mathring{\nu}\Delta$

note $\exists$ normal bundle $\Rightarrow$ tub. nbd.

HW: $\mathbb{Z} \cong H_1(M_K) \xrightarrow{\cong} H_1(W)$

$H_2(W) = 0$

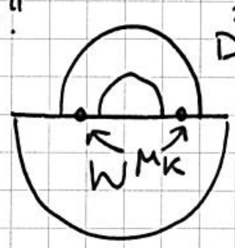
$\pi_1 W$ norm gen by μ_K.

Why is $\partial W = M_K$?

$\nu\Delta \cong D^2 \times D^2$. $\partial W = (S^3 \setminus \mathring{\nu} K) \cup (\underbrace{\Delta \times \partial D^2}_{\text{solid torus}})$.

note $\lambda_K \longleftrightarrow \partial\Delta$ by lk number.

($\Longleftarrow$) "Think of W as a slice disc exterior"



$M_K = S^3 \setminus \mathring{\nu} K \cup \underbrace{D^2 \times S^1}_{\text{glue here}}$

$\mathcal{B} := W \cup D^2 \times D^2$

$\mu_K \longleftrightarrow \partial D^2 \times D^2$

$\cap$

M_K

Note: $\pi_1 \mathcal{B} = 1$ since $\pi_1 W \cong \langle\langle \mu_K \rangle\rangle$.

Mayer-Vietoris $\Rightarrow$ $H_* \mathcal{B} = 0$.

Harder to see: $\partial \mathcal{B} \cong S^3$.

Reality check: μ_K bounds a disc by construction

4D Poincaré $\Rightarrow$ $\mathcal{B} \approx B^4$.

Also hard to see: $K \subseteq S^3 = \partial B^4$ is $\overset{\text{TOP}}{\text{slice}}$ via $\underbrace{* \times D^2}_{\text{in the original Dehn surgery solid torus.}} \subseteq D^2 \times D^2$

[but follows from the construction]